

Solve $\begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & 3 & 7 & -2 \\ 6 & -6 & 5 & -2 \\ 0 & -6 & -4 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ -6 \end{bmatrix}$

(LU Decomp.) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & 3 & 7 & -2 \\ 6 & -6 & 5 & -2 \\ 0 & -6 & -4 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & -6 & -4 & 1 \\ 0 & -6 & -4 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

Check: $-2 \stackrel{??}{=} 0 + 0 + 1 - 3$ ok.
 $5 \stackrel{??}{=} 9 - 2 - 2 + 0$ ok.

(Divide by L) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ -6 \end{bmatrix}$

$\begin{cases} a = 1 \\ b = 4 - 2 = 2 \\ c = 3 - 3 + 4 = 4 \\ d = -6 + 4 - 4 = -6 \end{cases}$

(Divide by U) $\begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -6 \end{bmatrix}$

$\begin{cases} -3z = -6 & z = 2 \\ -2y = 4 - 2 & y = -1 \\ 3x = 2 - (-1) & x = 1 \\ 2w = 1 + 3 + 2 & w = 3 \end{cases}$

Ans $\begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}$

Check:

$\begin{bmatrix} 1 \\ 4 \\ 3 \\ -6 \end{bmatrix} \stackrel{??}{=} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & 3 & 7 & -2 \\ 6 & -6 & 5 & -2 \\ 0 & -6 & -4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 - 3 - 2 \\ 12 + 3 - 7 - 4 \\ 18 - 6 - 5 - 4 \\ -6 + 4 - 4 \end{bmatrix}$ ok.

Solve $\begin{bmatrix} 3 & 1 \\ -6 & -3 \\ 9 & 2 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -3 \\ 9 \end{bmatrix}$

(LU decomp.) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -6 & -3 \\ 9 & 2 \\ -3 & -4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ *Rows of all 0 in U!*

(Divide by L) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -3 \\ 9 \end{bmatrix}$
 $\rightarrow a = 0$
 $\rightarrow b = -3 + 0 = -3$
 $\rightarrow c = -3 + 0 + 3 = 0$
 $\rightarrow d = 9 + 0 - 9 + 0 = 0$

(Divide by U) $\begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}$
 $\rightarrow \begin{cases} 3x + y = 0 \\ -y = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$

Answer: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Note: Because of the rows of all 0 in U, there would have been a problem if c or d $\neq 0$.

Solve $\begin{bmatrix} 2 & -3 \\ -4 & 1 \\ -2 & -7 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \\ 4 \end{bmatrix}$

(LU decomp) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 1 \\ -2 & -7 \\ 6 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & -5 \\ 0 & -10 \\ 0 & 15 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(Divide by L) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \\ 4 \end{bmatrix}$
 $\rightarrow a = 3$
 $\rightarrow b = -4 + 6 = 2$
 $\rightarrow c = 1 + 3 - 4 = 0$
 $\rightarrow d = 4 - 9 + 6 = 1$

(Divide by U) $\begin{bmatrix} 2 & -3 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$
 $\rightarrow \begin{cases} 2x - 3y = 3 \\ -5y = 2 \\ 0 = 0 \\ 0 = 1 \end{cases}$

Impossible!!

Ans: No Solution!

(Row of zero in U) \leftrightarrow (May be no solution)

Solve $\begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -2 & 6 \\ 3 & 5 & -10 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ -9 \end{bmatrix}$

(LU decomp)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -2 & 6 \\ 3 & 5 & -10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 4 & 0 \\ 0 & 4 & -8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(Divide by L)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ -9 \end{bmatrix}$$

$\rightarrow a = 2$
 $\rightarrow b = 10 - 6 = 4$
 $\rightarrow c = -9 - 2 + 8 = -3$

(Divide by U)

$$\begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$

column with no pivot

$\rightarrow 3z = -3$
 $\hookrightarrow z = -1$
 $-2x + 4y = 4$
 $\hookrightarrow y = y$ (free)
 $x = -2 + 2y$
 $3w + x - 2y + 2z = 2$
 $\hookrightarrow w = 2$

Ans: $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 + 2y \\ y \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} y$

The previous example did not have a unique solution. One variable was left "free" (undetermined) because there were 4 variables but only

3 equations. " $\begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} y$ " is a

(parameterized) line.

- \rightarrow Through the point $\begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \end{bmatrix}$
- \rightarrow In direction $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

Different parameterizations of this line give different ways of writing the same solution.

EX " $\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} x$ " is also soln.

The first part of the solution — the "point"

is a particular solution $\begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -2 & 6 \\ 3 & 5 & -4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ -9 \end{bmatrix}$

The second part of the solution — the "direction"

is a homogeneous solution $\begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -2 & 6 \\ 3 & 5 & -4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solve $\begin{bmatrix} 3 & -6 & -4 \\ 6 & -12 & -5 \\ -6 & 12 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -14 \end{bmatrix}$

(LU Decomp.) $\rightarrow = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & -4 \\ 6 & -12 & -5 \\ -6 & 12 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & -4 \\ 0 & 0 & 3 \\ 0 & 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & -4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

One zero row and one column with no pivot!

(Divide by L) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -14 \end{bmatrix}$

$\rightarrow a = -5$
 $\rightarrow b = -4 + 10 = 6$
 $\rightarrow c = -14 - 10 + 24 = 0$

(Divide by U) $\begin{bmatrix} 3 & -6 & -4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \\ 0 \end{bmatrix}$

$\rightarrow 3z = 6 \rightarrow z = 2$
 $\rightarrow 0 = 0$ ok.
 $\rightarrow 3x - 6y - 4z = -5$
 $3x - 6y - 8 = -5$
 $3x - 6y = 3$
 $x - 2y = 1$
 $\rightarrow y = y$ (free)
 $x = 1 + 2y$

(no pivot in "y" column...)

Ans: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+2y \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} y$

Solve $\begin{bmatrix} 2 & -3 & 1 \\ -6 & 9 & 1 \\ -8 & 12 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}$

(LU Decomp.) $\rightarrow = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ -6 & 9 & 1 \\ -8 & 12 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

One zero row and one column with no pivot!

(Divide by L) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}$

$\rightarrow a = 1$
 $\rightarrow b = -3 + 3 = 0$
 $\rightarrow c = -3 + 4 = 1$

(Divide by U) $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\rightarrow 0 = 1$ No!

(no pivot in "y" column)

No need to look at any other equations... This has no solution.

Ans: No Solution!